Homotopy Perturbation Method for Higher Dimensional Nonlinear Evolutionary Equations

Emanullah Hızel[a] and Semih Küçükarslan[b]

[a] Department of Mathematics, İstanbul Technical University, İstanbul, Turkey
[b] Department of Engineering Sciences, İstanbul Technical University, İstanbul, Turkey

Reprint requests to S. K.; Fax: +90-212-285 6386; E-mail: kucukarslan@itu.edu.tr

Z. Naturforsch. **64a,** 568 – 574 (2009); received September 22, 2008 / revised December 2, 2008

In this paper, an iterative numerical solution of the higher-dimensional (3+1) physically important nonlinear evolutionary equations is studied by using the homotopy perturbation method (HPM). For this purpose, the Kadomstev-Petviashvili (KP) and the Jumbo-Miwa (JM) equations are analyzed with the HPM and the available exact solutions obtained by the homogenous balance method will be compared to show the accuracy of the proposed numerical algorithm. The results approves the effectiveness and accuracy of the HPM.

Key words: Homotopy Perturbation Method (HPM); Kadomstev-Petviashvili (KP) Equation; Jumbo-Miwa (JM) Equation.

1. Introduction

In the last two decades, an important effort has been devoted to the solution of the higher dimensional, i. e. (2+1) and (3+1) dimensional, nonlinear partial differential equations. To obtain the solutions of such equations, various methods have been introduced. Mainly, it is worth to mention the Hirota method [1], the Lie Group analysis method [2 – 4], the travelling reduction applications [5], the tanh method [6, 7], the inverse scattering transform method [8], and the homogenous balance method [9 – 11].

For the solution of (3+1) dimensional nonlinear equations, the homogenous balance method is successfully applied to the Kadomstev-Petviashvili (KP) and the Jumbo-Miwa (JM) equations by Senthilvelan [12] and exact solutions were obtained.

It is important to obtain exact or numerical solutions in many fields of sciences and engineering, especially solid state physics, fluid mechanics, and plasma physics. Such as in plasma physics, the dusty plasmas are the interest of study since they exist in the universe almost everywhere. As a result nonlinear studies are strongly encouraging. For this reason, the nonlinear propagation of dust-ion-acoustic waves in an unmagnetized dust plasma was studied by considering a non-planar spherical geometry [13 – 16] for the KP and for the cylindrical KP model [17]. Another hierarchy of the KP model is called the JM model [12]. Because it is difficult or sometimes impossible to obtain an exact solution for these highly nonlinear partial differential equations, a powerful and validated method becomes very important. An iterative numerical method called homotopy perturbation method (HPM) was proposed by He [18] and further developed by him [19 – 22] for better accuracy. It yields a fast convergence for most of the selected problems. It also showed a high accuracy and a rapid convergence to solutions of the nonlinear partial differential equations.

Recently, homotopy perturbation technique successfully has been applied to (2+1) dimensional coupled Burgers system by Hızel and Küçükarslan [23].

In this current work, a higher dimensional (3+1) nonlinear evolutionary differential equation will be considered the first time by using the HPM. Since the selected equations, the KP and the JM equations, have analytical solutions, it will be illustrative to see the accuracy and the effectiveness of the HPM.

2. Homotopy Perturbation Method (HPM)

One considers the following nonlinear differential equation to represent the procedure of this method:

$$Au - f(r) = 0, \quad r \in \Omega, \tag{1}$$

with the boundary conditions

$$B\left(u,\frac{\partial u}{\partial n}\right)=0,\quad r\in\Gamma, \tag{2}$$

where A and B are the general differential operator and the boundary operator, respectively. Γ is the boundary of the domain Ω, and $f(r)$ is a given analytical function.

After dividing the general operator into linear part (L) and nonlinear part (N), one can rewrite (1) as

$$Lu+Nu-f(r)=0. \tag{3}$$

By constructing the homotopy technique to (3), one can write a homotopy in the form

$$H(v,p)=(1-p)[L(v)-L(u_0)]+p[A(v)-f(r)]=0,\quad p\in[0,1],\ r\in\Omega, \tag{4}$$

where p is an embedding parameter and u_0 an initial approximation of (1) which satisfies (2).

In the HPM, one can use the embedding parameter as a small parameter. Therefore, the solution of (4) can be written as a power series of p in the following form:

$$v=v_0+pv_1+p^2v_2+\dots. \tag{5}$$

By setting $p=1$, one can get an approximate solution of (1):

$$u=\lim_{p\to 1}v=v_0+v_1+v_2+\dots. \tag{6}$$

The combination of a small parameter (perturbation parameter) with a homotopy is called homotopy perturbation method as presented in the final equation (6).

3. Application of the HPM to (3+1) Dimensional Nonlinear PDE(s)

3.1. The Jumbo-Miwa (JM) Equation

The governing equation for the JM is given by

$$u_{xxxy}+3u_{xy}u_x+3u_yu_{xx}+2u_{yt}-3u_{xz}=0. \tag{7}$$

The exact solution of (7) was obtained by using the homogenous balance method [12] in the following form:

$$u=a_0+2k\alpha\tanh[k(\alpha x+\beta y+\gamma z-\lambda t)], \tag{8}$$

where k, α, β, γ are arbitrary constants with $a_0=2k\alpha$ and $\lambda=2k^2\alpha^3-\frac{3\alpha\gamma}{2\beta}$.

By assigning $k=1$, $\alpha=1$, $\gamma=2$, $\beta=3$, one obtains $a_0=2$, $\lambda=1$, and (8) is simplified to

$$u=2+2\tanh[x+3y+2z-t]. \tag{9}$$

By choosing for (9) the initial condition $u_0=u_0(0,y,0,t)=2+2\tanh[3y-t]$, substituting (9) into (4), and then substituting (5), one gets a system of equations with $n+1$ terms which need to be solved simultaneously. Since computations are dependent on the value of u_0, a minor modification gives flexibility to choose the initial u_0. For this purpose, the following homotopy is constructed as Refai et al. [24] propose:

$$\begin{aligned}&(1-p)(3v_{xz}-3u_{0xz})\\&+p(-v_{xxxy}-3v_{xy}v_x-3v_yv_{xx}-2v_{yt}+3v_{xz})=0.\end{aligned} \tag{10}$$

After expanding (10) for $n=3$, i. e. the third power of p, one obtains

$$\begin{aligned}&p^0: v_{0xz}=u_{0xz},\\&p^1: 3v_{1xz}-v_{0xxxy}-2v_{0yt}-3v_{0xy}v_{0x}-3v_{0y}v_{0xx}=0,\\&p^2: 3v_{2xz}-v_{1xxxy}-2v_{1yt}-3v_{0xy}v_{1x}-3v_{1xy}v_{0x}\\&\quad-3v_{1y}v_{0x}-3v_{1xx}v_{0y}=0,\\&p^3: 3v_{3xz}-v_{2xxxy}-2v_{2yt}-3v_{0xy}v_{2x}-3v_{1xy}v_{1x}\\&\quad-3v_{0x}v_{2xy}-3v_{0y}v_{2xx}-3v_{1y}v_{1xx}-3v_{2y}v_{0xx}=0.\end{aligned} \tag{11}$$

The solution of (11) can be obtained as:

$$\begin{aligned}v_0&=2+2\tanh(3y-t),\\v_1&=-8xz\,\mathrm{sech}^2(t-3y)\tanh(t-3y),\\v_2&=4x^2z^2\,\mathrm{sech}^5(t-3y)(\sinh(3t-9y)\\&\quad-11\sinh(t-3y)),\\v_3&=\frac{-16}{3}z^3\left\{\frac{1}{3}x^3\Big[123+\cosh(4t-12y)\right.\\&\quad-56\cosh(2t-6y)\\&\quad\left.-18x\big(-3+\cosh(2t-6y)\big)\Big]\right\}\\&\quad\cdot\,\mathrm{sech}^6(t-3y)\tanh(t-3y).\end{aligned} \tag{12}$$

By substituting these values into (6), one can get an approximation to u:

$$\begin{aligned}&u\cong v_0+v_1+v_2+v_3=\\&2+2\tanh(3y-t)-8xz\,\mathrm{sech}^2(t-3y)\tanh(t-3y)\\&+4x^2z^2\,\mathrm{sech}^5(t-3y)[\sinh(3t-9y)-11\sinh(t-3y)]\end{aligned}$$

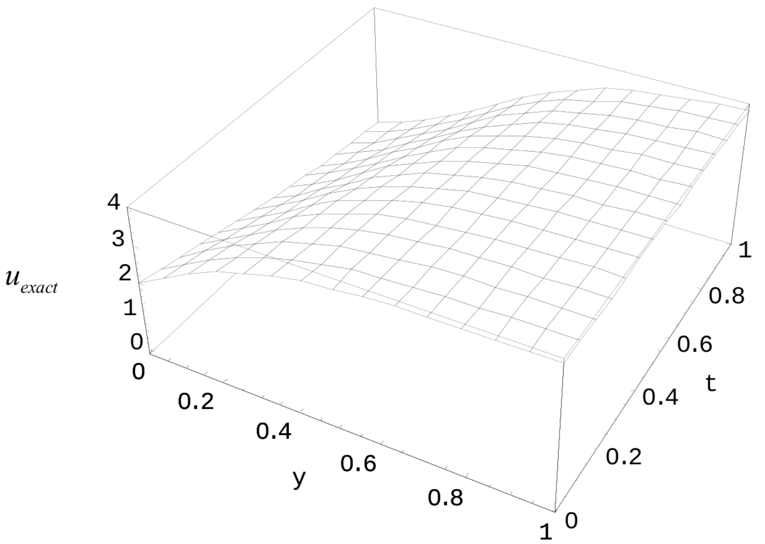

Fig. 1. Exact solution of $u(0,y,0,t)$ for $0 \le y \le 1$ and $0 \le t \le 1$.

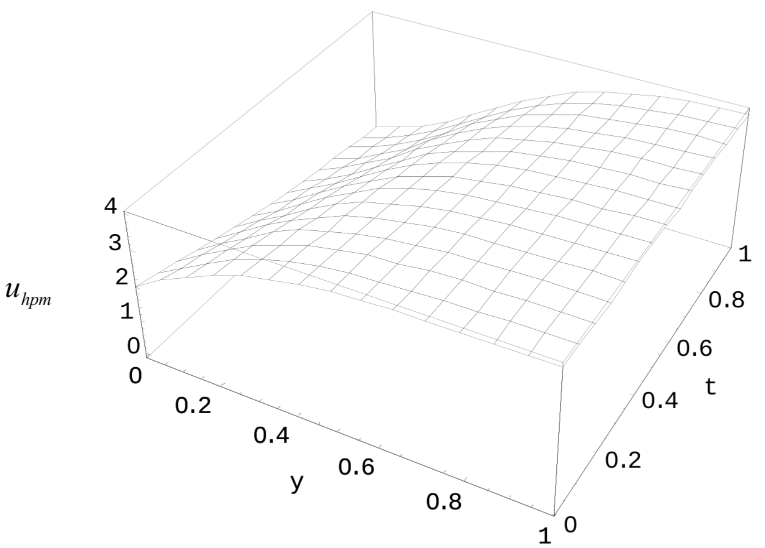

Fig. 2. HPM solution of $u(0,y,0,t) \cong v_0+v_1+v_2+v_3$ for $0 \le y \le 1$ and $0 \le t \le 1$.

$$
\begin{aligned}
-\frac{16}{3}z^3\Big\{\frac{1}{3}x^3\Big[&123+\cosh(4t-12y)\\
&-56\cosh(2t-6y)\\
&-18x\big(-3+\cosh(2t-6y)\big)\Big]\Big\}\\
&\cdot\operatorname{sech}^6(t-3y)\tanh(t-3y). \qquad (13)
\end{aligned}
$$

In Figure 1, the exact solution of $u(x,y,z,t)$ for the intervals $0 \le y \le 1$ and $0 \le t \le 1$ and for $x=0$, $z=0$ is plotted. The approximated solution of $u(x,y,z,t)$ is plotted for the first four terms in Figure 2 in the same interval.

3.2. The Kadomstev-Petviashvili (KP) Equation

The governing equation for the KP is given by

$$u_{xt}+6u_x^2+6uu_{xx}-u_{xxxx}-u_{yy}-u_{zz}=0. \qquad (14)$$

The exact solution of (14) was obtained by using the homogenous balance method [12] by the following form:

$$
\begin{aligned}
u=\left(\frac{-4k^2\alpha^2}{3}+\frac{2k^2\alpha^2}{3}\right)&\\
+\,2k^2\alpha^2\tanh^2[k(\alpha x+\beta y+\gamma z-\lambda t)],& \qquad (15)
\end{aligned}
$$

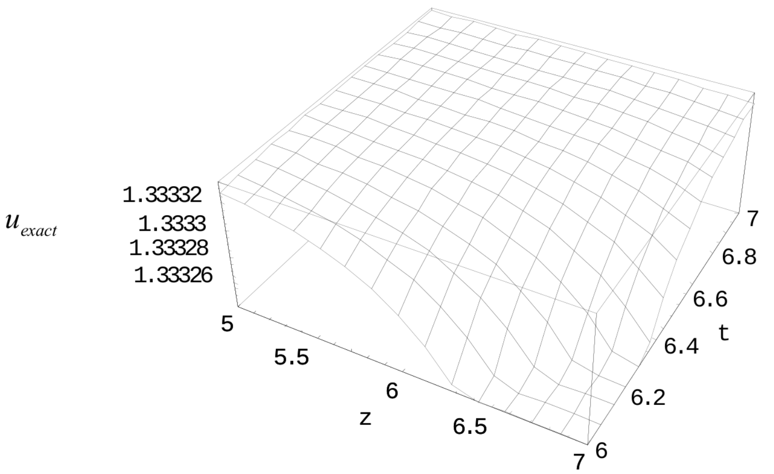

Fig. 3. Exact solution of $u(0,0,z,t)$ for $5 \le z \le 7$ and $6 \le t \le 7$.

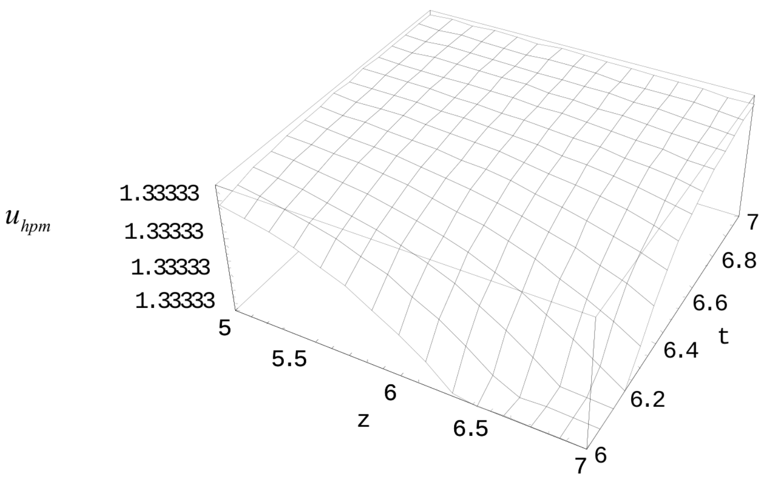

Fig. 4. HPM solution of $u(0,0,z,t) \cong v_0+v_1+v_2+v_3+v_4$ for $5 \le z \le 7$ and $6 \le t \le 7$.

where k, α, β, γ are arbitrary constants with $\lambda = \frac{-(\beta^2+\gamma^2)+4k^2\alpha^2}{\alpha}$. By assigning $k=1$, $\alpha=1$, $\gamma=1$, and $\beta=1$, one obtains $\lambda=2$ and (15) is simplified to

$$u = -2/3 + 2\tanh^2[x+y+z-2t]. \tag{16}$$

By selecting an initial u_0 for $z=0$, one obtains

$$u_0 = u_0(x,y,0,t) = -2/3 + 2\tanh^2[x+y-2t]. \tag{17}$$

By following the same procedure as in the JM case, one can construct the homotopy in the form

$$\begin{aligned}&(1-p)(v_{zz}-u_{0zz})\\&+p(-v_{xt}-6v_x^2-6vv_{xx}+v_{xxxx}+v_{yy}+v_{zz})=0.\end{aligned} \tag{18}$$

After expanding (18) for $n=2$, i. e. the second power of p, one obtains

$$\begin{aligned}&p^0: v_{0zz} = u_{0zz},\\&p^1: v_{1zz}+v_{0yy}+v_{0xxxx}-v_{0xt}-6v_{0x}^2-6v_{0xx}v_0=0,\\&p^2: v_{2zz}+v_{1yy}+v_{1xxxx}-v_{1xt}-12v_{0x}v_{1x}-6v_{0xx}v_1\\&\quad -6v_{1xx}v_0=0.\end{aligned} \tag{19}$$

The solution of (19) can be obtained as:

$$\begin{aligned}v_0 &= \frac{-2}{3}+2\tanh^2(x+y-2t),\\v_1 &= 2z^2\{-2+\cosh[4t-2(x+y)]\}\mathrm{sech}^4(2t-x-y),\\v_2 &= \frac{-1}{24}z^4\{1960+\cosh[12t-6(x+y)]\\&\quad +168\cosh[8t-4(x+y)]\\&\quad +1905\cosh[4t-2(x+y)]\}\mathrm{sech}^8(2t-x-y).\end{aligned} \tag{20}$$

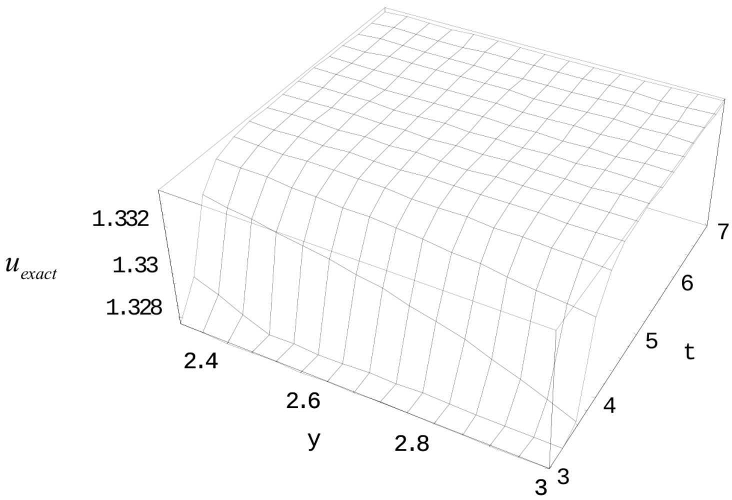

Fig. 5. Exact solution of $u(0,y,0.5,t)$ for $2.3 \le y \le 3$ and $3 \le t \le 7$.

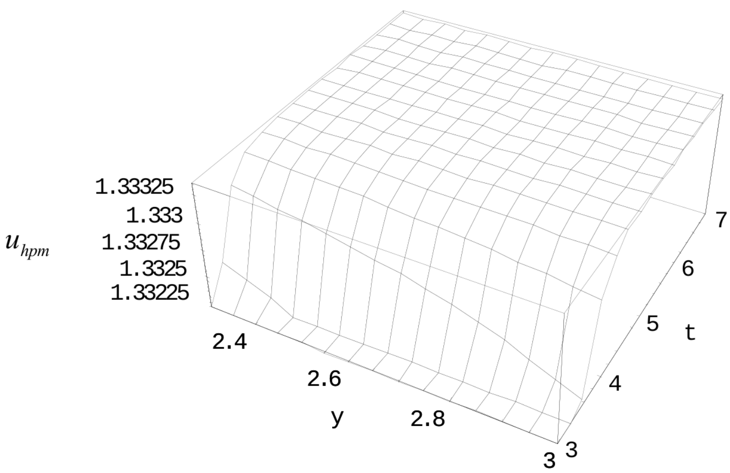

Fig. 6. HPM solution of $u(0,y,0.5,t) \cong v_0+v_1+v_2+v_3+v_4$ for $2.3 \le y \le 3$ and $3 \le t \le 7$.

By substituting these values into (6), one can get an approximation for u:

$$
\begin{aligned}
&u \cong v_0+v_1+v_2 = \\
&-\frac{2}{3}+2\tanh^2(x+y-2t) \\
&+2z^2\big\{-2+\cosh[4t-2(x+y)]\big\}\operatorname{sech}^4(2t-x-y) \\
&-\frac{1}{24}z^4\big\{1960+\cosh(12t-6(x+y)] \\
&+168\cosh(8t-4(x+y)))\operatorname{sech}^8(2t-x-y) \\
&-\frac{1}{24}z^4 1905\cosh[4t-2(x+y)]\big\}\operatorname{sech}^8(2t-x-y).
\end{aligned}
\tag{21}
$$

In Figure 3, the exact distribution of $u(x,y,z,t)$ in the intervals $5 \le z \le 7$ and $6 \le t \le 7$ as well as $x=0$ and $y=0$ is plotted. The approximated distribution of $u(x,y,z,t)$ is plotted for the first five terms in Figure 4 in the same intervals.

In Figure 5, the exact distribution of $u(x,y,z,t)$ in the intervals $2.3 \le y \le 3$ and $3 \le t \le 7$ as well as $x=0$ and $z=0.5$ is plotted. The approximated distribution of $u(x,y,z,t)$ is plotted for the first five terms in Figure 6 in the same intervals.

In Figure 7, the exact distribution of $u(x,y,z,t)$ in the intervals $2.3 \le x \le 3$ and $0 \le t \le 1$ as well as $y=3$ and $z=0.5$ is plotted. The approximated distribution of $u(x,y,z,t)$ is plotted for the first five terms in Figure 8 in the same intervals.

4. Conclusions

In this paper, an iterative numerical solution of a physically interesting problem of the higher dimen-

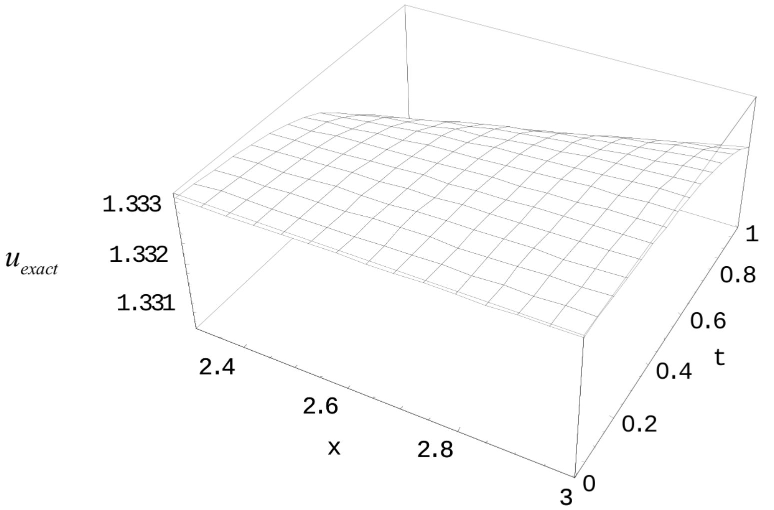

Fig. 7. Exact solution of $u(x,3,0.5,t)$ for $2.3 \le x \le 3$ and $0 \le t \le 1$.

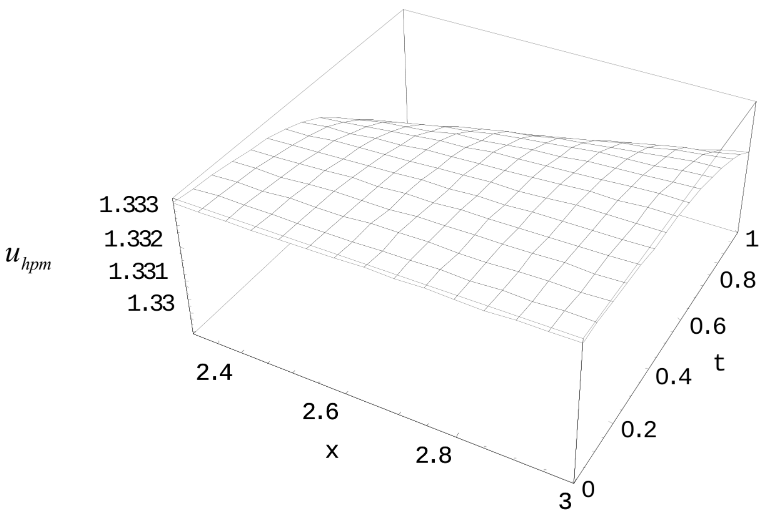

Fig. 8. HPM solution of $u(x,3,0.5,t) \cong v_0 + v_1 + v_2 + v_3 + v_4$ for $2.3 \le x \le 3$ and $0 \le t \le 1$.

sional (3+1) evolutionary nonlinear partial differential equations, the KP and the JM equations, was studied by using the homotopy perturbation method and the obtained results were compared with the available exact solutions. These solutions will be helpful to understand the physics of dusty plasmas, dust-ion-acoustic waves for industrial and physical applications. It can be concluded that He's HPM gives a very good approximation and a fast convergence to the (3+1) dimensional nonlinear PDEs.

[1] M. Laksmanan and R. Radha, Pramana-J. Phys. **48**, 163 (1997).
[2] G. Paguin and P. Winternitz, Physica D **46**, 122 (1990).
[3] M. Faucher and P. Winternitz, Phys. Rev. E **48**, 3066 (1993).
[4] M. Senthilvelan and M. Laksmanan, J. Nonlin. Math. Phys. **5**, 190 (1998).
[5] Z. Yan and H. Zhang, Phys. Lett. A **252**, 291 (1999).
[6] Y. Tian Gao and B. Tian, Chaos, Solitons, and Fractals **8**, 897 (1997).

[7] B. Tian and Y. Tian Gao, Comput. Phys. Commun. **95**, 139 (1996).
[8] M. J. Ablowitz and P. A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering Transform, Cambridge University Press, Cambridge 1990.
[9] M. L. Wang, Phys. Lett. A **199**, 169 (1995).
[10] M. L. Wang, Phys. Lett. A **213**, 279 (1996).
[11] M. L. Wang, Y. B. Zhou, and Z. B. Li, Phys. Lett. A **216**, 67 (1996).
[12] M. Senthilvelan, App. Math. and Comp. **123**, 381 (2001).
[13] Y. Tian Gao and B. Tian, Phys. Lett. A **340**, 243 (2005).
[14] Y. Tian Gao and B. Tian, Phys. Lett. A **361**, 523 (2007).
[15] Y. Tian Gao and B. Tian, Phys. Plasmas **13**, 222901 (2006).
[16] Y. Tian Gao and B. Tian, Europyhsics Lett. **77**, 15001 (2007).
[17] Y. Tian Gao and B. Tian, Phys. Lett. A **362**, 283 (2007).
[18] J. H. He, Commun. Nonlin. Sci. Numer. Simul. **3**, 92 (1998).
[19] J. H. He, Comput. Methods Appl. Mech. Eng. **178**, 257 (1999).
[20] J. H. He, Int. J. Nonlin. Mech. **35**, 37 (2000).
[21] J. H. He, Appl. Math. Comput. **135**, 73 (2003).
[22] J. H. He, Appl. Math. Comput. **156**, 527 (2004).
[23] E. Hızel and S. Küçükarslan, Nonlin. Analysis B **10**, 1932 (2009).
[24] M. Refai, D. D. Ganji, H. R. M. Daniali, and H. Pashaei, Phys. Lett. A **364**, 1 (2007).